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Tracer Diffusion in Lattice Gases

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It has been proved that a tracer particle in a reversible lattice gas converges to Brownian motion. However, only in a few particular cases has a strictly positive self-diffusion coefficient D been established. Here we supply the missing piece and show that $D > 0$ in general. The exceptions are one-dimensional lattice gases with nearest neighbor jumps only, for which $D = 0$. The proof establishes a variational formula for D which could be used to obtain realistic bounds.

KEY WORDS: Stochastic lattice gases; self-diffusion; variational formula.

1. INTRODUCTION

Tracer diffusion is a standard problem of nonequilibrium statistical mechanics: one considers a large system of interacting particles in thermal equilibrium (=fluid) and adds one extra particle (=tracer particle, test particle, tagged particle). The tracer particle is distinguished through some internal property, e.g., color, from the fluid particles. Let $x_t \in \mathbb{R}^d$ be the position of the tracer particle at time t. One expects that on a large space-time scale, x_t behaves as Brownian motion. The immediate and best known example is pollen immersed in water. (At that time the experimental observation of Brownian motion was striking evidence for the atomistic structure of matter.[1]) The phenomenon of Brownian motion is of a more general nature, however. In particular, the tracer particle does not have to be heavy in comparison to the fluid particles.

We investigate here tracer diffusion in the context of stochastic lattice gases (see ref. 2 for a review). When going to a larger scale, length and time

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must be related through $(\text{length})^2 \approx \text{time}$ as appropriate for Brownian motion. Therefore, the rescaled position of the tracer particle is defined as

$$x_t^\varepsilon = \varepsilon x_{\varepsilon^{-2}t} \tag{1.1}$$

ε small. It has been proved,[3,4] under fairly mild conditions, that

$$\lim_{\varepsilon \to 0} x_t^\varepsilon = (2D)^{1/2}\, b(t) \tag{1.2}$$

Here D is the self-diffusion coefficient (better, self-diffusion matrix) defined through

$$\langle (l \cdot x_t)^2 \rangle = 2(l \cdot Dl)\, t \tag{1.3}$$

for large t with l a constant vector and $l \cdot x = \sum_{\alpha=1}^{d} l_\alpha x_\alpha$. $b(t)$ is standard Brownian motion in d dimensions. This is the Gaussian process with mean zero and covariance

$$\mathbb{E}(b_\alpha(t)\, b_\beta(s)) = \delta_{\alpha\beta} \min(t, s) \tag{1.4}$$

α, $\beta = 1,..., d$. In contrast, for mechanical systems governed by Newton's equations of motion, the convergence to Brownian motion has been established only in a few very special cases.

We consider lattice gases with the constraint of single site occupancy. For such systems the tracer particle may be blocked by a surrounding large cluster of lattice gas particles. The tracer particle may have to wait then, possibly a long time, until the cluster has changed to the point where nearby sites open up. In fact, in one dimension with nearest neighbor jumps only, blocking is so severe that $D = 0$.[5] The leading long-time behavior is not Brownian motion, but subdiffusive. In more than one dimension blocking should not have such a drastic effect. Our aim is to prove that, with the exception already noted, indeed $D > 0$ always.

Such a result has been established in the case of infinite temperature (the reversible stationary measures are Bernoulli).[3] A simple lower bound for D is obtained by freezing all lattice gas particles. This attempt fails at densities above the percolation threshold. The lower bound vanishes then because the tracer particle is confined to a finite number of sites.

To give a short outline: In Section 2 we define the stochastic lattice gas and the tracer particle motion. In Section 3 we establish a variational formula for the self-diffusion coefficient. This formula is novel, to our knowledge, and was inspired by the corresponding one for bulk diffusion which I learned from S. R. S. Varadhan. The variational formula is used to obtain a lower bound on the self-diffusion coefficient. The basic idea is simple. First we constrain the tracer particle motion to one lattice axis. To avoid blocking, a lattice gas particle adjacent to the tracer particle along the specified axis may jump across the tracer particle. All other lattice gas particles are frozen. We allow just enough motion of the lattice gas to avoid blocking and so little that a positive self-diffusion coefficient can be shown by elementary means.

2. CONVERGENCE TO BROWNIAN MOTION

We consider a standard reversible lattice gas on $\mathbb{Z}^d$.[6] The state space is $X = \{0, 1\}^{\mathbb{Z}^d}$. A configuration of particles is denoted by $\zeta \in X$. Here $\zeta(x)$, $x \in \mathbb{Z}^d$, is the occupation variable at site x: $\zeta(x) = 1$ if the site x is occupied and $\zeta(x) = 0$ if site x is vacant. ζ^{xy} denotes the configuration with occupancies at x and y interchanged,

$$\zeta^{xy}(u) = \begin{cases} \zeta(y) & \text{if} \quad u = x \\ \zeta(x) & \text{if} \quad u = y \\ \zeta(u) & \text{if} \quad u \neq x, y \end{cases} \tag{2.1}$$

Let $c(x, y, \zeta)$ be the exchange rate (jump rate) between x and y. It has to satisfy:

(i) $c(x, y, \zeta) = c(y, x, \zeta) \geqslant 0$.

(ii) (translation invariance)

$$c(x + a, y + a, \tau_a \zeta) = c(x, y, \zeta)$$

where τ_a is the shift by $a \in \mathbb{Z}^d$.

(iii) (finite range). There exists a constant R such that

$$c(x, y, \zeta) = 0$$

if $|x - y| > R$ and such that $c(x, y, \zeta)$ depends only on the $\zeta(u)$'s with $|u - y| \leqslant R$, $|u - x| \leqslant R$.

(iv) (nondegeneracy)

$$c(x, y, \zeta) > 0$$

for $|x - y| = 1$ and $\zeta(x) \neq \zeta(y)$.

To impose reversibility, we first need an interaction energy, which is constructed from a set of potentials $\{J_A \mid A \text{ finite subset of } \mathbb{Z}^d\}$. The potential is translation invariant, $J_{A+a} = J_A$, and of finite range, i.e., $J_A = 0$, whenever the distance between two points in A exceeds R. The energy H_Λ in the bounded subset Λ is given by

$$H_\Lambda(\zeta) = \sum_{A \subset \Lambda} J_A \left(\prod_{x \in A} \zeta(x) \right) \tag{2.2}$$

Note that, because of the finite range, energy differences are defined even in infinite volume,

$$H(\zeta^{xy}) - H(\zeta) = \lim_{\Lambda \uparrow \mathbb{Z}^d} [H_\Lambda(\zeta^{xy}) - H_\Lambda(\zeta)] \tag{2.3}$$

Reversibility is imposed through the following condition.

(v) (detailed balance)

$$c(x, y, \zeta) = c(x, y, \zeta^{xy}) \exp\{-[H(\zeta^{xy}) - H(\zeta)]\} \tag{2.4}$$

The generator of the lattice gas dynamics is

$$L_0 f(\zeta) = \frac{1}{2} \sum_{x, y} c(x, y, \zeta)[f(\zeta^{xy}) - f(\zeta)] \tag{2.5}$$

acting on local functions f. The semigroup generated by L_0 is Feller and the stochastic jump process ζ_t is constructed in the standard fashion.[6]

Any Gibbs measure μ for the potential (J_A) is stationary for ζ_t.[7]. We require here the following condition.

(vi) μ is extreme translation invariant.

Actually, it suffices that μ is extreme invariant under a lattice subgroup of all translations.

The initial measure for ζ_t is the Gibbs measure μ conditioned that $\zeta(0) = 1$. [Under our assumptions $\mu(\zeta(0)) > 0$.] The particle initially at the origin is our tracer particle. Its position at time t is denoted by x_t.

It turns out to be convenient to adopt a moving frame of reference by choosing the position of the tracer particle as the origin. Then the configuration as seen from the tracer particle is

$$\eta_t(x) = \zeta_t(x + x_t) \tag{2.6}$$

By definition,

$$\eta_t(0) = 1 \tag{2.7}$$

for all $t \geqslant 0$. Therefore the η-configuration space is

$$X_0 = \{\eta \mid \eta(0) = 1\}$$

The generator of the lattice gas dynamics as seen from the tracer particle has two parts, L_1 and L_2. The part L_1 describes the shifting of the whole configuration due to jumps of the tracer particle,

$$L_1 f(\eta) = \sum_{x, x \neq 0} c(0, x, \eta)[1-\eta(x)][f(\tau_{-x}\eta^{0x}) - f(\eta)] \tag{2.8}$$

for local functions f on X_0. The part L_2 describes the jumping of particles with the origin forbidden, since occupied,

$$L_2 f(\eta) = \frac{1}{2} \sum_{x, y, x \neq 0 \neq y} c(x, y, \eta)[f(\eta^{xy}) - f(\eta)] \tag{2.9}$$

for local functions f on X_0. The generator of the full dynamics is then

$$L = L_1 + L_2 \tag{2.10}$$

Let μ_0 be the Gibbs measure μ conditioned on $\eta(0) = 1$. Then μ_0 is reversible under both L_1 and L_2.

So far the tracer particle has been dynamically identical to all other particles, in the sense that its jumps are governed by the same jump rates. This does not need to be the case. If not, we redefine L_1 as

$$L_1 f(\eta) = \sum_{x, x \neq 0} \tilde{c}(0, x, \eta)[1-\eta(x)][f(\tau_{-x}\eta^{0x}) - f(\eta)] \tag{2.11}$$

The jump rates $\tilde{c}(0, x, \eta)$ have to satisfy $\tilde{c}(0, x, \eta) \geqslant 0$ and conditions (iii) and (iv). Furthermore, μ_0 has to be reversible for L_1. This is ensured by the following condition.

(v′) (detailed balance)

$$\begin{aligned}\tilde{c}(0, x, \eta)(1-\eta(x)) = {}& \tilde{c}(0, -x, \tau_{-x}\eta^{0x})(1 - \tau_{-x}\eta^{0x}) \\ & \times \exp\{-[H(\eta^{0x}) - H(\eta)]\}\end{aligned} \tag{2.12}$$

η_t is the stationary process governed by $L = L_1 + L_2$ with μ_0 as initial measure. Clearly, given $\eta_{t'}$ for $0 \leqslant t' \leqslant t$ we can reconstruct the position x_t of the tracer particle.

Next we want to determine the self-diffusion coefficient D. For this we consider the joint process (x_t, η_t) with generator L_g. Now, $x_0 = 0$ and η_0 is distributed according to μ_0. Let $f(x, \eta) = l \cdot x$ with l some real vector. Then

$$L_g f(x, \eta) = j_l(\eta) \tag{2.13}$$

with

$$j_l(\eta) = \sum_{x, x \neq 0} \tilde{c}(0, x, \eta)[1-\eta(x)](l \cdot x) \tag{2.14}$$

By standard Markov theory we have the martingale

$$M_t = l \cdot x_t - \int_0^t ds\, j_l(\eta_s) \tag{2.15}$$

Its quadratic variation is

$$\begin{aligned} \mathbb{E}(M_t^2) &= t \sum_{x, x \neq 0} (l \cdot x)^2 \langle \tilde{c}(0, x, \eta)[1 - \eta(x)] \rangle_0 \\ &= \mathbb{E}\left(\left[(l \cdot x_t) - \int_0^t ds\, j_l(\eta_s) \right]^2 \right) \end{aligned} \tag{2.16}$$

Here $\langle \cdot \rangle_0$ is expectation with respect to μ_0. Working out the square, the cross term vanishes because x_t is odd and $\int_0^t ds\, j_l(\eta_s)$ is even under time reversal. Thus,

$$\begin{aligned} \frac{1}{t} \mathbb{E}((l \cdot x_t)^2) = & \sum_{x, x \neq 0} (l \cdot x)^2 \langle \tilde{c}(0, x, \eta)[1 - \eta(x)] \rangle_0 \\ & - \frac{1}{t} \int_0^t ds \int_0^t ds' \langle j_l e^{L|s - s'|} j_l \rangle_0 \end{aligned} \tag{2.17}$$

η_t is a reversible process. Therefore, by the spectral theorem the limit $t \to \infty$ in (2.17) exists and defines the self-diffusion matrix

$$\begin{aligned} D_{\alpha\beta} = & \frac{1}{2} \delta_{\alpha\beta} \sum_x x_\alpha^2 \langle \tilde{c}(0, x, \eta)[1 - \eta(x)] \rangle_0 \\ & - \int_0^\infty dt\, \langle j_\alpha e^{Lt} j_\beta \rangle_0 \end{aligned} \tag{2.18}$$

Here $j_\alpha = j_{e_\alpha}$ with e_α the unit vector in the positive α direction. $D \geqslant 0$ as a $d \times d$ matrix. In principle, both terms in (2.18) could cancel each other.

Theorem 1.[3,4] Let the above assumptions hold and let D be defined by (2.18). Then

$$\lim_{\varepsilon \to 0} \varepsilon x_{\varepsilon^{-2} t} = (2D)^{1/2}\, b(t) \tag{2.19}$$

$b(t)$ is d-dimensional standard Brownian motion. The convergence is in the sense of weak convergence of path measures on $D([0, \infty), \mathbb{R}^d)$.

Before closing this section, we briefly comment on the velocity autocorrelation function of the tracer particle. We define the velocity of the tracer particle by

$$v_t = \frac{d}{dt} x_t \tag{2.20}$$

in the distributional sense. v_t is a stationary stochastic process. Comparing with (2.17), its autocorrelation is given by

$$\mathbb{E}(v_t \cdot v_0) = \delta(t) \sum_x x^2 \langle \tilde{c}(0, x, \eta)[1 - \eta(x)] \rangle_0 - \sum_{\alpha=1}^{d} \langle j_\alpha e^{L|t|} j_\alpha \rangle_0 \tag{2.21}$$

The velocity autocorrelation has a δ peak at $t = 0$ and is otherwise negative. It increases monotonically ($t \geqslant 0$) and is integrable. The velocity autocorrelation function has a universal feature known as a long-time tail.[8,9] This refers to its asymptotic decay, which is expected to be of the form

$$\mathbb{E}(v_t \cdot v_0) \approx -t^{-[(d/2)+1]} \tag{2.22}$$

for large t. Of course, the prefactor of the long-time tail will depend on the particular jump rates.

For symmetric exclusion in one dimension with nearest neighbor jumps Arratia[5] proves that $\mathbb{E}(x_t^2) \approx \sqrt{t}$ for large t and that $t^{-1/4} x_t$ has a Gaussian distribution asymptotically. Transferring the first result to the velocity autocorrelation gives a $-t^{-3/2}$ decay in accordance with the theoretical prediction. In fact, this is one of the few cases for which a long-time tail in the velocity autocorrelation has been proved.[9,10]

3. A VARIATIONAL FORMULA

Since η_t is a reversible process, the self-diffusion coefficient can also be expressed in variational form.

Proposition 2. Let D be given by (2.18). Then

$$(l \cdot Dl) = \frac{1}{2} \inf \left\{ \sum_{x, x \neq 0} \langle \tilde{c}(0, x, \eta)[1 - \eta(x)][l \cdot x + f(\tau_{-x}\eta^{0x}) - f(\eta)]^2 \rangle_0 + \frac{1}{2} \sum_{x, y, x \neq 0 \neq y} \langle c(x, y, \eta)[f(\eta^{xy}) - f(\eta)]^2 \rangle_0 \right\} \tag{3.1}$$

The infimum is over all real-valued local functions f on X_0.

Proof. Since $\int_0^\infty dt \, \langle j_l e^{Lt} j_l \rangle_0 < \infty$, we have $(-L)^{1/2} j_l \in L^2(X_0, \mu_0)$. Therefore

$$2\langle j_l L^{-1} j_l \rangle_0 = \inf_f (-4\langle j_l f \rangle_0 - 2\langle fLf \rangle_0) \tag{3.2}$$

The Dirichlet forms are

$$-2\langle fL_1 f\rangle_0 = \sum_{x,x\neq 0} \langle \tilde{c}(0,x,\eta)[1-\eta(x)][f(\tau_{-x}\eta^{0x})-f(\eta)]^2\rangle_0 \tag{3.3}$$

$$-2\langle fL_2 f\rangle_0 = \frac{1}{2}\sum_{x,y,x\neq 0\neq y} \langle c(x,y,\eta)[f(\eta^{xy})-f(\eta)]^2\rangle_0 \tag{3.4}$$

Using the detailed balance (2.12), we have

$$\begin{aligned} -4\langle j_l f\rangle_0 &= -4\sum_{x,x\neq 0} (l\cdot x)\langle \tilde{c}(0,x,\eta)[1-\eta(x)]\,f(\eta)\rangle_0 \\ &= 2\sum_{x,x\neq 0} (l\cdot x)\langle \tilde{c}(0,x,\eta)[1-\eta(x)][f(\tau_{-x}\eta^{0x})-f(\eta)]\rangle_0 \end{aligned} \tag{3.5}$$

Inserting (3.5), (3.3), and (3.4) in (3.2) and adding the first term of (2.18) results in (3.1). ▮

As an immediate consequence of the variational formula, we note that D is increasing (as a matrix) in c and $\tilde{c}$.

4. A LOWER BOUND

We prove that (2.19) is indeed the true long-time behavior.

Theorem 3. Let the above assumptions hold and let D be given by (2.18).

(i) If $d=1$ and

$$c(x, x+2, \eta) > 0 \tag{4.1}$$

for $\eta(x)\neq\eta(x+2)$, then $D>0$.

(ii) If $d>1$, then $D>0$ as a matrix.

We want to reduce the lower bound in higher dimensions to a lower bound in one dimension. We single out the 1 axis and set $e=(1,0,\dots,0)$. Of course, the final result will be independent of this choice. The jumps of the tracer particle will be restricted to the 1 axis. Lattice gas particles away from the tracer particle will be frozen. We first establish a lower bound to the Dirichlet form (3.4) consisting of jumps between $-e$ and e.

Lemma 4. Let $d\geqslant 2$. We label the sites of a path with nearest neighbor bonds from e to $-e$ by $x=1,2,\dots,5$. Let

$$L_{xx+1}f(\eta) = c(x,x+1,\eta)[f(\eta^{xx+1})-f(\eta)] \tag{4.2}$$

$$L_{15}f(\eta) = \bar{c}(1,5,\eta)[f(\eta^{15})-f(\eta)] \tag{4.3}$$

There exists a choice of the reversible exchange rate $\bar{c}(1, 5, \eta)$ such that $\bar{c}(1, 5, \eta) > 0$ for $\eta(1) \neq \eta(5)$ and such that

$$-\langle f L_{15} f \rangle_0 \leqslant -\sum_{x=1}^{4} \langle f L_{xx+1} f \rangle_0 \tag{4.4}$$

for all local functions f.

Proof. Let $T_{ij} f(\eta) = f(\eta^{ij})$, $i, j = 1,..., 5$. Then

$$T_{15} = T_{12} T_{23} T_{34} T_{45} T_{34} T_{23} T_{12} \equiv T T_{12} \tag{4.5}$$

By telescoping,

$$T_{15} f - f = T(T_{12} f - f) + \cdots + (T_{12} f - f) \tag{4.6}$$

and by the Schwarz inequality,

$$\tfrac{1}{7}(T_{15} f - f)^2 \leqslant T(T_{12} f - f)^2 + \cdots + (T_{12} f - f)^2 \tag{4.7}$$

Now, with c a suitable positive constant,

$$c T(T_{12} f - f)^2 \leqslant T[\exp(TH - H)\, c(1, 2)(T_{12} f - f)^2] \tag{4.8}$$

because $c(1, 2, \eta) > 0$ for $\eta(1) \neq \eta(2)$. Therefore we can choose a positive exchange rate $\bar{c}(1, 5)$ such that

$$\begin{aligned} 2\bar{c}(1, 5)(T_{15} f - f)^2 \leqslant{}& T[\exp(TH - H)\, c(1, 2)(T_{12} f - f)^2] + \cdots \\ &+ c(1, 2)(T_{12} f - f)^2 \end{aligned} \tag{4.9}$$

We average in (4.9) over $\langle \cdot \rangle_0$ and use that the dual of T_{ij} is given by

$$\langle g T_{ij} f \rangle_0 = \langle \exp[-(T_{ij} H - H)](T_{ij} g) f \rangle_0 \tag{4.10}$$

The result is (4.4). ∎

We ignore terms in the variational formula (3.1) and use Lemma 4. Then

$$\begin{aligned} l_1^{-2}(l \cdot D l) \geqslant \frac{1}{2} \inf_f \Big\{ & \sum_{x = \pm e} \langle \tilde{c}(0, x, \eta)[1 - \eta(x)] \\ & \times [(e \cdot x) + f(\tau_{-x} \eta^{0x}) - f(\eta)]^2 \rangle_0 \\ & + \langle \bar{c}(-e, e, \eta)[f(\eta^{-ee}) - f(\eta)]^2 \rangle_0 \Big\} \\ \equiv \bar{D} & \end{aligned} \tag{4.11}$$

$\bar{D}$ is again the diffusion coefficient of a tracer particle. We describe its motion in the lattice fixed frame: Let ζ_t be the lattice gas configuration and y_t be the position of the tracer particle at time t. The position y_t is included in the lattice gas configuration; hence $\zeta_t(y_t)=1$. Initially, $y_t=0$ and $\zeta=\zeta_0$ is distributed according to the conditional Gibbs measure μ_0. y_t jumps by $\pm e$. If $\zeta(-e)=1=\zeta(e)$, then $y_t=0$. We disregard this case and assume $\zeta(-e)\,\zeta(e)=0$, which has a nonzero probability for μ_0. If the site y_t+e is vacant, then y_t jumps there with rate $\tilde{c}(0, e, \tau_{-y_t}\zeta_t)$, and correspondingly for $-e$. The tracer particle does not jump to occupied sites. A lattice gas particle can move only if it is adjacent, with $\pm e$, to the tracer particle. Otherwise the configuration remains frozen. If $\zeta_t(y_t+e)=1$ and, consequently, $\zeta_t(y_t-e)=0$, then with rate $\bar{c}(-e, e, \tau_{-y_t}\zeta_t)$ the configuration ζ_t changes to $\bar{\zeta}$, where $\bar{\zeta}(x)=\zeta_t(x)$ for $x\neq y_t\pm e$ and $\bar{\zeta}(y_t+e)=0$, $\bar{\zeta}(y_t-e)=1$; and correspondingly for $-e$.

Lemma 5. Let y_t be the position of the tracer particle as defined above. There exists a strictly positive constant c such that

$$\mathbb{E}(y_t^2)\geqslant ct \tag{4.12}$$

Proof. The simple observation is that ζ_t (including the position of the tracer particle) can be thought of as a Markov chain with state space $\mathbb{Z}$ and nearest neighbor jumps only. A positive diffusion coefficient is easily established for it.

Let us denote the initial configuration by ζ^0. By assumption, $\zeta^0(-e)\,\zeta^0(e)=0$ and $\zeta^0(0)=1$. The configurations which can be reached from ζ^0 are labeled as ζ^n, $n\in\mathbb{Z}$, and are defined iteratively by the following rule: Let y be the position of the tracer particle in the configuration ζ^n. If $\zeta^n(y+e)=0$, then $\zeta^{n+1}=(\zeta^n)^{yy+e}$, and if $\zeta^n(y-e)=0$, then $\zeta^{n-1}=(\zeta^n)^{y-ey}$. If $\zeta^n(y+e)=1$ and therefore by construction $\zeta^n(y-e)=0$, then $\zeta^{n+1}=(\zeta^n)^{y-ey+e}$. Correspondingly, if $\zeta^n(y-e)=1$, and therefore $\zeta^n(y+e)=0$, then $\zeta^{n-1}=(\zeta^n)^{y-ey+e}$. Let $n(t)$ be the jump process with generator

$$Gf(n)=\lambda_+(n)[f(n+1)-f(n)]+\lambda_-(n)[f(n-1)-f(n)] \tag{4.13}$$

We can choose the jump rates $\lambda_+(n)$, $\lambda_-(n)$ such that

$$\zeta_t=\zeta^{n(t)} \tag{4.14}$$

By construction, there exist constants c_+, c_- independent of ζ^0 such that

$$0<c_-\leqslant\lambda_+(n),\qquad \lambda_-(n)\leqslant c_+<\infty \tag{4.15}$$

and the rates satisfy detailed balance in the form

$$\lambda_+(n) = \lambda_-(n+1)\exp\{-[H(\zeta^{n+1}) - H(\zeta^n)]\} \tag{4.16}$$

The energy differences are bounded from below and above independent of ζ^0. Most importantly, we have

$$4y^2 \geqslant n^2 \tag{4.17}$$

because y moves by one unit at least every second step.

We construct a harmonic function, $Gf = 0$, which is essentially linear. Using detailed balance, we obtain

$$f(n) = \begin{cases} \displaystyle\sum_{z=0}^{n-1} \frac{1}{\lambda_+(z)} \exp[H(\zeta^z) - H(\zeta^0)] & \text{for} \quad n \geqslant 1 \\ 0 & \text{for} \quad n = 0 \\ \displaystyle -\sum_{z=n+1}^{0} \frac{1}{\lambda_-(z)} \exp[H(\zeta^z) - H(\zeta^0)] & \text{for} \quad n \leqslant -1 \end{cases} \tag{4.18}$$

Furthermore,

$$Gf^2(n) = [\lambda_+(n)^{-1} + \lambda_-(n)^{-1}]\exp\{2[H(\zeta^n) - H(\zeta^0)]\} \tag{4.19}$$

Therefore, we can choose constants c_1 and c_2 independent of ζ^0 such that

$$\frac{d}{dt}\mathbb{E}(f(n(t))^2) \geqslant c_1 \tag{4.20}$$

$$f(n)^2 \leqslant c_2 n^2 \tag{4.21}$$

Together with (4.17), this implies

$$\mathbb{E}(y_t^2) \geqslant \tfrac{1}{4}\mathbb{E}(n(t)^2) \geqslant (c_2/4)\,\mathbb{E}(f(n(t))^2) \geqslant (c_1 c_2/4)t \quad \blacksquare \tag{4.22}$$

Proof of Theorem 3. Part (i) follows from Lemma 5. For part (ii), Lemmas 4 and 5 together establish

$$(l \cdot Dl) \geqslant c l_1^2 \tag{4.23}$$

with c some positive constant. Repeating the same argument for the other coordinate axis yields $(l \cdot Dl) \geqslant \tilde{c} l_\alpha^2$ with $\tilde{c} > 0$ and $\alpha = 1,..., d$. This proves our assertion. ▌

We supplement Theorem 3 by the following result.

Proposition 6. Let $d=1$ and let

$$c(0, x, \eta)=0=\tilde{c}(0, x, \eta) \tag{4.24}$$

for $|x| \geqslant 2$. Then $D=0$.

Proof. We use

$$f(\eta)=\sum_x h(\varepsilon x)\, \eta(x) \tag{4.25}$$

as variational function in (3.1). h is a function of compact support on $\mathbb{R}$, which is smooth except at zero, where $\rho(h(0_+)-h(0_-))=-1$ with ρ the average density for μ. Then

$$\begin{aligned} D \leqslant \sum_{e=\pm 1} \Big\langle & \tilde{c}(0, e)[1-\eta(e)] \\ & \times \Big\{e+\sum_{x, x \neq 0, e}[h(\varepsilon x-\varepsilon e)-h(\varepsilon x)]\, \eta(x)\Big\}^2\Big\rangle_0 \\ & +\sum_{x, x \neq -1,0} \langle c(x, x+1, \eta)[\eta(x)-\eta(x+1)]^2 \\ & \times [h(\varepsilon x)-h(\varepsilon x+\varepsilon)]^2\rangle_0 \end{aligned} \tag{4.26}$$

For small ε, the second term is of order ε. The first term becomes

$$\sum_{e=\pm 1}\Big\langle\tilde{c}(0, e)[1-\eta(e)]\Big[1-\varepsilon \sum_{x \neq 0, e} h'(\varepsilon x)\, \eta(x)\Big]^2\Big\rangle_0 \tag{4.27}$$

We use the ergodic theorem and the exponentially fast approach to the average density as $|x| \rightarrow \infty$ to conclude that this term also vanishes as $\varepsilon \rightarrow 0$. ∎

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